A constitutive equation for concrete

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A more rational approach to strength criterion development for concrete is proposed to cover the composite nature and complex failure mechanism of concrete materials. The use of scalar valued function theory as applied to concrete failure prediction is demonstrated. The results are applicable to general brittle materials

1. Introduction

The characteristic properties of concrete have been shown to be those of a complex, multi-phase material which is best studied as a composite. The physical properties in the final state (hydrated state) depend on the original mixed proportions and the environmental conditions during cure. Real concrete is, in general, non-homogeneous, anisotropic, and non-continuous, as it is composed of groups of elements formed into a large number of discrete particles. However, there is a dimensional level of aggregation (the phenomenological or engineering level) at which the concept of the structural element can be replaced by a homogeneous, isotropic, continuous medium composed of structural elements of identical properties. The mechanical characteristics of concrete are best idealized at the macroscopic level for engineering design applications. The assumption of homogeneity can be justified only on a statistical basis after consideration of the average properties of the elements in the body.

The failure for concrete has been shown to be initiated by numerous microscopic flaws or cracks inherent within the concrete matrix. The average influence of these microscopic flaws, as viewed from macroscopic theory, reveal distinct levels of change in the mechanical behaviour of concrete. As the stress level increases, the mechanical behaviour changes from quasi-elastic to plastic, with two distinct points of departure. The initial discontinuity begins at the onset of stable fracture propagation while the ultimate strength is reached at the onset of unstable fracture propagation. The hydrostatic (spherical) and deviatoric components of the localized stress have been shown to delay and propagate the internal crack growth, respectively.

The development of a strength criterion for a material depends on its stress state at or during failure conditions; either it is brittle or ductile. Consideration of mechanical response and failure mode shows that concrete is best classified as a brittle material for normal hydrostatic pressure. The strength characterization of most brittle materials depends on the hydrostatic as well as the deviatoric component of stress, while the characterization of the ductile material is independent of the hydrostatic component.

Most strength criteria presented in previous papers follow functional form, which are functions of stress tensors. The strength criteria presented in these papers show poor agreement with experimental results. The theories presented often require given material property co-ordinate systems, and are not invariant, and so require complex methodologies for characterization of material parameters. These criteria have, for the most part, been formulated within the framework of classical theories of plasticity which are subject to a number of strong constraints. These approaches lack generality and agreement with physical laws.

In recent years, as complex, anisotropic, fibre-reinforced composites have been developed, more appropriate methods for the characteristics of materials have been sought. In the field of non-linear continuum mechanics there have been continuous developments following more powerful approaches to these problems. In reviewing the recently proposed general strength criteria, the continuum mechanics approach has been most prominent. The application of general and explicit tensor based scalar-valued or tensor-valued functions has proven to be highly useful for developing strength criteria and constitutive equations. Many investigations have shown the value of using tensor function theory in these applications.

The composite nature and complex failure mechanism of concrete dictate a need for a more rational approach to strength criterion development. The purpose of this study is to demonstrate the utility of a scalar valued function theory as applied to concrete failure prediction. The general results are applicable to any quasi-elastic brittle material, but for the purpose of concrete characterization a specific strength criterion for concrete is developed.

2. Development of the Proposed Strength Criterion

The development of a strength criterion for the prediction of ultimate strength of concrete under multiaxial loadings should be formulated from the systematic theories of modern continuum mechanics. The criterion should be validated by accurate experimental data for the determination of the failure surface for

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concrete. A strength criterion to predict the failure of concrete is by necessity governed by the failure mechanisms. These failure mechanisms must be related mathematically, and yield a failure surface in a stress-space.

The tensor functional technique of non-linear continuum mechanics is the most logically applicable to the formulation of the strength criterion. Such functionals satisfy the requirement of invariance for a group of orthogonal transformations specific to the material symmetry. In addition, tensor function theory allows inclusion of any number of stress interaction terms, which gives the theory a broad applicability to the characterization of anisotropic material. The tensor functional technique for the development of a strength criterion is a new approach which produces a rational criterion.

A strength function has been shown to be expressible as

$$f(\sigma_{ij}) = 1 \quad i,j, = 1,2,3 \tag{1}$$

where σ_{ij} is a stress tensor referred to an arbitrary coordinate system. The form of the failure function in Equation 1 has been followed by past investigators. In general, the strength criteria presented were functions of the applied stress which were non-invariant, i.e. William and Warnke [1], Wastiels [2, 3], Kotsovos [4], and others.

A strength function for a given material symmetry (isotropic for concrete) must be invariant under a complete point group of transformations of coordinates, $\{t_{ij}\}$, which associate with the group of material symmetry. This ensures that the strength criterion is a scalar (invariant under the appropriate group of coordinate transformation), and is a single-valued function, as indicated by Equation 1. It is known that failure is a physical phenomenon which is totally independent of co-ordinates. Thus the requirement of invariance states

$$f(\bar{\sigma}_{ij}) = f(\sigma_{ij}) \quad i,j = 1,2,3 \tag{2}$$

where $(\bar{\sigma}_{ij})$ represents the transformed stress components

$$\bar{\sigma}_{ij} = t_{ir}\, t_{js}\, \sigma_{rs} \quad i,j,r,s = 1,2,3 \tag{3}$$

Invariant quantities for each class of anisotropic materials have been obtained by Smith and Rivlin [5], and Huang [6]. Huang determined the second, fourth, and sixth order of invariant quantities in the three-dimensional case for each of the crystal classes from consideration of invariant transformations of the strength function. The invariant quantities for the isotropic material symmetry case are as follows

$$\begin{aligned} I_1^{(1)} &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2^{(2)} &= -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ I_3^{(3)} &= \sigma_1\,\sigma_2\,\sigma_3 \end{aligned} \tag{4}$$

where $I_j^{(i)}$ is the j-th invariant quantities of i-th degree.

The strength function for an isotropic material in the form of Equation 1 can be rewritten as

$$f(I_1, I_2, I_3) = 1 \tag{5}$$

Also, Equation 5 can be expressed in terms of the deviatoric and spherical invariant quantities

$$\begin{aligned} J_2 &= 1/6[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2] \\ J_3 &= (\sigma_1-\sigma)(\sigma_2-\sigma)(\sigma_3-\sigma) \end{aligned} \tag{6}$$

where

$$\sigma = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3) = \frac{1}{3}I_1$$

Therefore, a strength function is also expressible as a function

$$f(I_1, J_2, J_3) = 1 \tag{7}$$

The proposed strength function by Chen and Chen [7] followed the invariant of a tensor function of a second degree. A two-equation strength criterion, using the invariant quantities of Equations 7, is given as

$$f(I_1, J_2) = J_2 + \frac{A_u}{3} I_1 = t_u^2 \tag{8a}$$

for the compression–compression region, and for all other regions, (compression–tension, tension–tension, and tension–compression) as

$$f(I_1, J_2) = J_2 - \frac{1}{6}I_1^2 + \frac{A_u}{3}I_1 = t_u^2 \tag{8b}$$

where A_u and t_u^2 are material parameters.

The strength functions of Equation 8 are a special form of linear combination of invariants. The functions are of quadratic form. The quadratic form has been addressed and shown to be inadequate in its definition of the failure envelope for the biaxial principal stress plane. The cubical forms of the polynomial based on tensor function theory were discussed by Huang [6, 8]. Tennyson *et al.* [9], Ottosen [10], and Priddy [11]. They suggested that the third degree terms are necessary to be included and to explain the additional stress interaction relations. The quadratic form at best can describe a conic curve which may not yield accurate correlations with experimental data for concrete in all four quadrants of the biaxial plane.

In order to formulate a cubic strength function, the invariant quantities of $I_j^{(3)}$ must be included. Thus the invariant quantities of each degree for an isotropic material are

$$\begin{gathered} I^{(1)},\ \text{first degree: } I_i \\ I^{(2)},\ \text{second degree: } I_1^2, J_2 \\ I^{(3)},\ \text{third degree: } I_1^3, I_1 J_2, J_3, I_1^2 J_2^{1/2} \end{gathered} \tag{9}$$

The system of quantities (Equation 9) represents terms which are required to form a cubic strength function for isotropic materials.

A strength function, which is a combination of the invariant quantities (Equation 9), has been proposed for concrete, as follows

$$f(I_j^{(i)} = A_1 I_1 + A_{11} I^2 + A_{22} J_2 + A_{111} I_1^3 + A_{122} I_1 J_2 + A_{333} J_3 \mp A_{112} I_1^2 J_2^{1/2} = 1 \tag{10}$$

where all As are material parameters of the strength tensor. These parameters are determined from experimental strength data for concrete.

This paper concerns a biaxial stress condition which causes failure. The proposed cubic equation (Equation 10) can be reduced from seven to six material parameters for a biaxial loading condition. Since the third degree of the deviator stress invariant (J_3) is a combination of three other invariants

$$J_3 = I_3 + \frac{1}{3} I_1 J_2 - \frac{1}{27} I_1^3 \tag{11}$$

and I_3 is zero for the case of the biaxial state of stress, the third degree deviator term J_3 can be eliminated. Thus Equation 10 is reduced to

$$A_1 I_1 + A_{11} I_1^2 + A_{22} J_2 + A_{111} I_1^3 + A_{122} I_1 J_2 \mp A_{112} I_1^2 J_2^{1/2} = 1 \tag{12}$$

The coefficient of the cubic term A_{112} in Equation 10 requires a sign change to completely characterize all four quadrants of the biaxial regions of stress. The sign of A_{112} is positive (+) in all quadrants, except the tension–tension quadrant where the sign is changed to be negative (−). This change was found necessary to allow for a close fit of the failure envelope in the tension–tension region to experimental data.

It should be noted that the strength function given by Chen and Chen [7] can be shown as a special case of the proposed cubic function of Equation 12.

3. The Strength Envelope Graph

Equation 12 gives the strength envelope for the biaxial states of stress. This envelope represents the strength surface on the σ_1–σ_2 principal co-ordinate plane.

The strength envelope of the biaxial states of stress must be closed to ensure the stability of the material. In other words, any radial line from the origin of co-ordinates (zero stress state) must intersect the strength envelope at one and only one point. A cubic strength function, as in Equation 12, with real coefficients has three roots on a loading path. At least one real root for Equation 12 exists. For this reason, adequate or appropriate weights must be assigned to those data points for which the strength coefficients (the As of Equation 12) are determined, so that the strength envelope forms a closed curve, which has only one intersection for any loading line.

Equation 12 cannot be solved analytically in a closed form. Alternatively, an iterative numerical analysis is developed using the Newton–Raphson technique. Let R denote the ratio of σ_2 to σ_1 or $R = \sigma_2/\sigma_1 = \tan\theta$, where θ represents the slope of a radial loading path.

With the iteration scheme Equation 12 can be rewritten as

$$F(\sigma_1) = E \tag{13}$$

where E denotes the residual. This equation yields

$$E = A\sigma_1^3 + B\sigma_1^2 + C\sigma_1 - 1 \tag{14}$$

where the coefficients A, B, and C are functions of the given values of A_i, A_{ij}, A_{ijk} and R.

If the correct value of σ_1 is substituted into Equation 14, the value of E will be zero. On the other hand, if an incorrect value of σ_1 is used, Equation 14 yields a non-zero value of residual. A Newton–Raphson technique is used to obtain the roots for σ_1. The correction value

$$\Delta\sigma_1 = -[E/(3A\sigma_1^2 + 2B\sigma_1 + C)] \tag{15}$$

is determined to improve the estimated value of σ_1. The new improved value of σ_1 is $(\sigma_1)_{n+1} = (\sigma_1)_n + (\Delta\sigma_1)_n$ where n denotes the number of iterations.

An initial estimate value of σ_1 is made. This can be done by setting σ_1 equal to $(\sigma_1)_0$, the value of the uniaxial compressive strength (for $R_0 = 0$). After having obtained a root, $(\sigma_1)_0$, which corresponds to $R = R_0 = 0$, the value of R can be perturbed, $R_1 = R_0 + \Delta R$. For this value of R_1, iteration is repeated starting from $\sigma_1 = (\sigma_1)_0$. If ΔR is not exceedingly large, then $(\sigma_1)_0$ is contained in the new contraction domain of the Newton method, and iteration converges quadratically to the root, $(\sigma_1)_1$, corresponding to $R = R_1$. Successive repetition of this analytical continuation approach leads to the solution for the strength envelope. The correct solution $(\sigma_1, \sigma_2)_{R=(n+1)\Delta R}$ for the current calculation is in the neighbourhood of the previous solution $(\sigma_1, \sigma_2)_{R=n\Delta R}$.

In order to ensure that a smooth and closed strength envelope is obtained, Equation 12 must yield three real roots. After one real root of Equation 12 is obtained, the other two real roots can be found as follows. By eliminating the first real root obtained (say σ) from the cubic equation, a quadratic equation is the result, i.e.

$$A_1\sigma_1^2 + B_1\sigma_1 + C_1 = 0 \tag{16}$$

where $A_1 = A$, $B_1 = B + A_1\sigma$, and $C_1 = C + B_1\sigma$. If coefficients A_1, B_1, and C_1 satisfy the inequality

$$B_1^2 - 4A_1C_1 \geqslant 0 \tag{17}$$

the two remaining roots of Equation 12 will be real. Otherwise the other two roots are conjugate complex. If the latter case occurs, the strength coefficients A_i, A_{ij} and A_{ijk} must be re-evaluated. However, the iterative process adopted in this paper usually proceeds smoothly, the re-examination of the other two roots to be real is not necessary. In this way the complete strength envelope, as expected, can be obtained easily on a graph.

TABLE I Six Material Coefficients

	Concrete compressive strength (MPa)		
	18.63	30.705	57.615
A_1	− 8.762	− 8.729	− 8.315
A_{11}	− 33.923	− 39.995	− 43.690
A_{22}	89.000	138.408	164.366
A_{111}	− 6.165	− 17.500	− 22.948
A_{122}	− 110.481	− 250.285	− 328.088
A_{112}	98.745	181.025	226.142

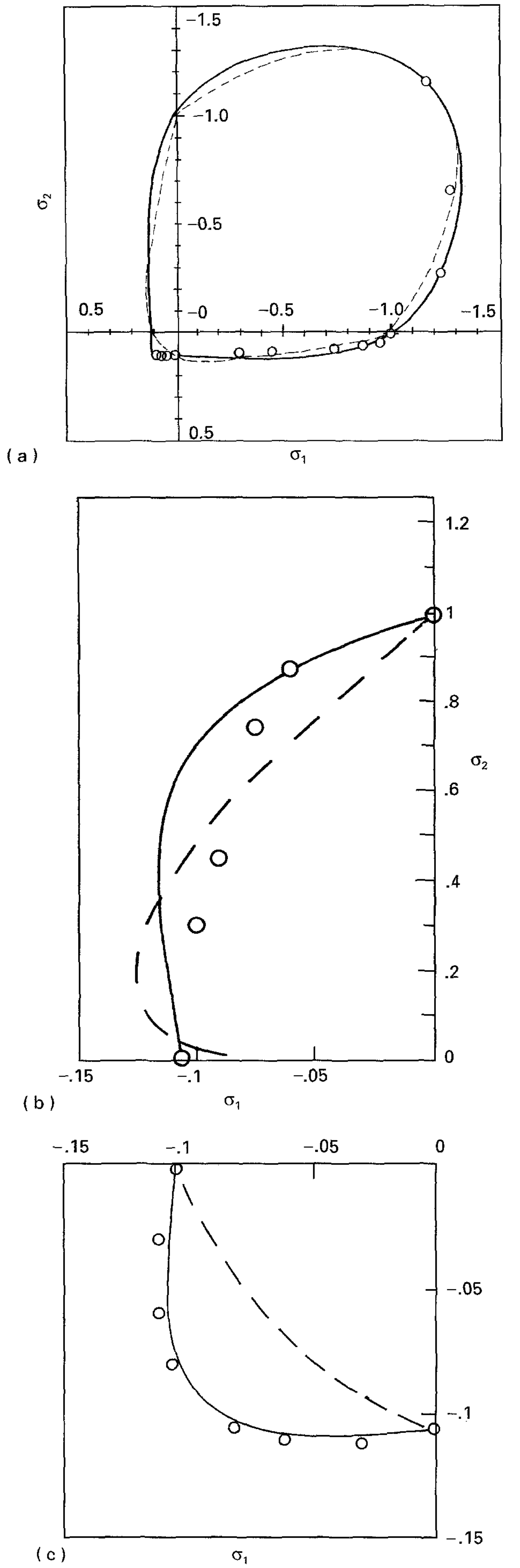

Figure 1 (a) Strength surface for concrete with $\sigma_c = 18.63$ MPa. $A_1 = -8.54$; $A_{11} = -27.45$; $A_{22} = +79.72$; $A_{111} = -5.66$; $A_{112} = +85.04$; $A_{122} = -99.08$. (----) Chen and Chen [7]; (——) cubic function. (b) Strength surface in C–T region. (c) Strength surface in C–C region.

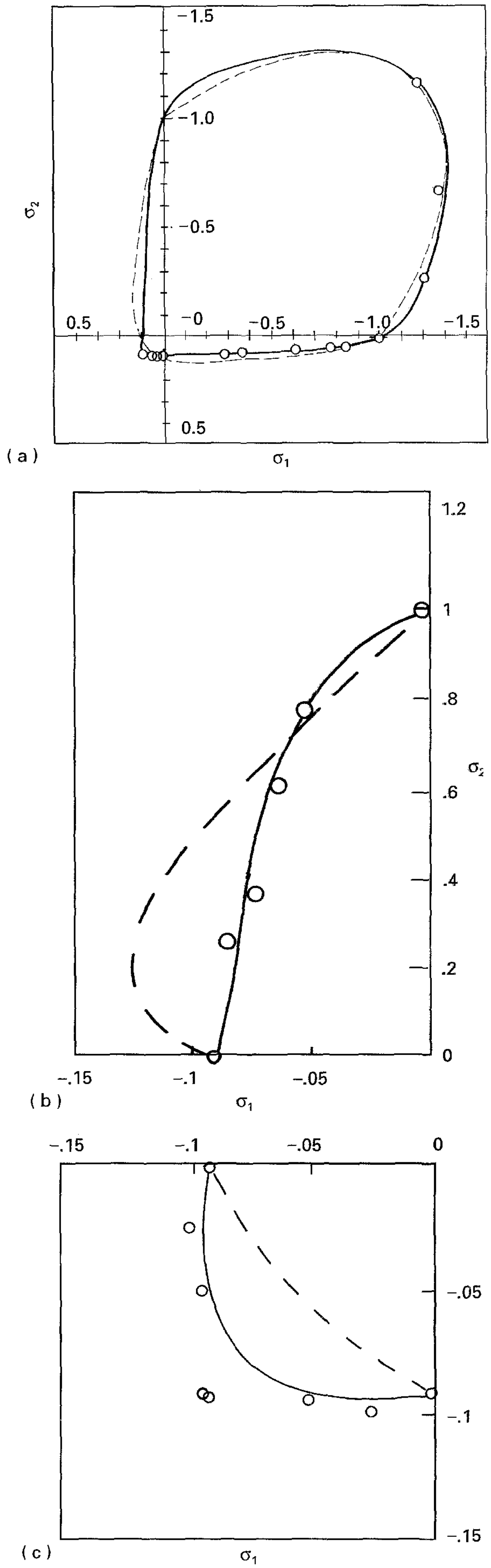

Figure 2 (a) Strength surface for concrete with $\sigma_c = 30.705$ MPa, $A_1 = -8.65$; $A_{11} = -33.22$; $A_{22} = +126.25$; $A_{111} = -15.98$; $A_{112} = +160.77$; $A_{122} = -228.13$. (----) Chen and Chen [7]; (——) cubic function. (b) Strength surface in C–T region. (c) Strength surface in C–C region.

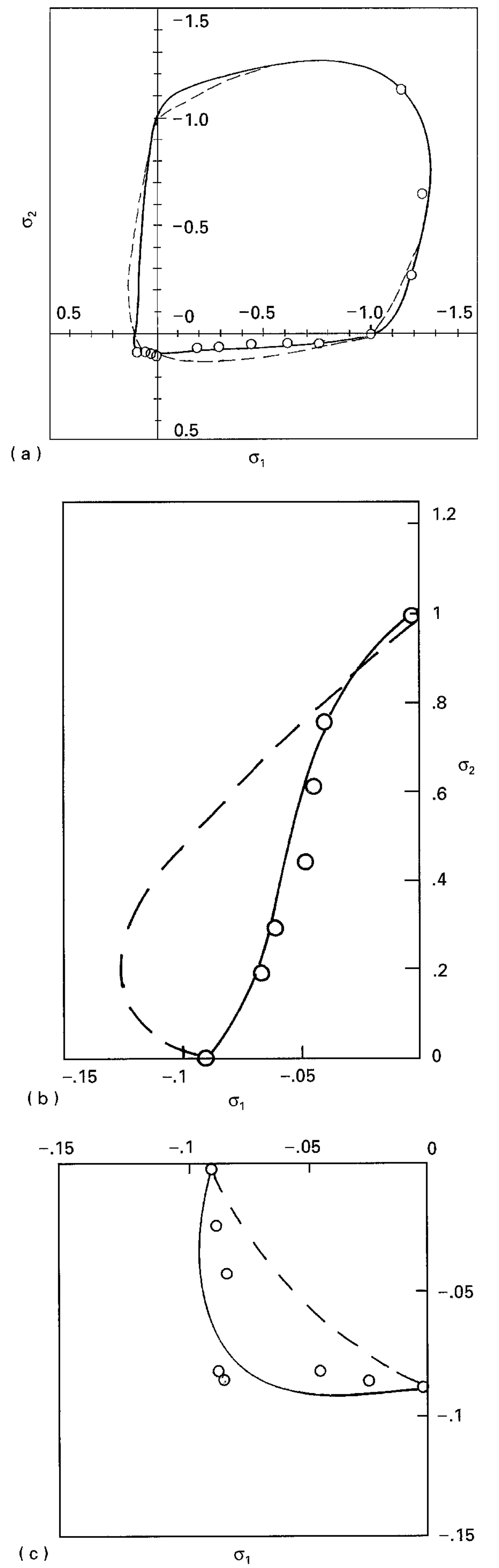

Figure 3 (a) Strength function for concrete with $\sigma_c = 57.615$ MPa, $A_1 = -8.51$; $A_{11} - 34.30$; $A_{22} = +141.72$; $A_{111} = -19.51$; $A_{112} = +188.40$; $A_{122} = -278.07$. (----) Chen and Chen [7]; (——) cubic function. (b) Strength surface in C–T region. (c) Strength surface in C–C region.

4. Numerical Examples

A computer program has been developed. Incorporating the determination of the strength coefficients and the real roots of the cubic equation, the best fit strength envelope is obtained. In order to verify the proposed strength criterion, the experimental data [12] for concrete are used for the calculations.

For the case of biaxial stress in the σ_1–σ_2 plane, the strength coefficients for three different compressive strengths obtained and summarized in Table I.

The strength envelopes are shown in Figs 1, 2 and 3. For the purpose of comparison, the results obtained by Chen and Chen [7] are given in figures. In particular the envelopes obtained by the proposed cubic strength criterion and the results from Chen and Chen, and the experimental data are plotted again in the quadrants (T–C) and (T–T) in Figs 1–3.

5. Conclusion

The new proposed cubic polynomial strength criterion (Equation 12) has been compared with the existing experimental strength data for the biaxial stress state. A favourable correlation between theoretical and experimental results is observed.

A systematic and straightforward numerical method of determining strength coefficients as well as the plotting of stress envelope have been developed. This method which is based on the concept of the stress tensor invariants, is also suitable for evaluation of strength criteria for other brittle materials as graphites and ceramics.

The configuration of the failure surface, obtained in this paper, agrees much better with experimental data than the others, particularly in the regions of T–T and C–T of stress space.

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